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Approximation by Taylor polynomials.

Recall the formula of Taylor polynomial:

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{E_n(x)}, \quad \xi \in (x, x_0) \text{ or } (x_0, x)$$

the approximation just means use the $P_n(x)$ (poly-nomial part) to estimate $f(x)$.

$f(x) \approx P_n(x)$, with the error $E_n(x)$.

Remark: 2 things should be considered:

(1) the point x_0. in order to compute the $P_n(x)$ easily, we have to choose the point x_0 which $f(x_0), f'(x_0) \cdots f^{(n)}(x_0)$ are easily obtained. And x_0 should be close to the point x we want to approximate in order to get higher accuracy;

(2). the order n. we can image that the higher order n we choose, the higher accuracy would be (in general), but of course the computation increase. So under some restrictions like we need the Error $|E_n(x)| < tol$ be controlled by some small number tol. Then we can get an inequality from $|E_n(x)| < tol$ that related to n. and we choose the smallest one as our order.

Q1. Try to estimate the constant e with error less than 10^{-6}.

Pf: we use e^x for our approximation.

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \frac{e^\xi}{(n+1)!}x^{n+1} \quad (x_0 = 0) \quad \xi \in (0, x))$$

$$\text{So} \Rightarrow e = e^1 = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + \frac{e^\xi}{(n+1)!} \quad \xi \in (0,1)$$

$$\Rightarrow |E_n(x)| = \left|\frac{e^\xi}{(n+1)!}\right| < tol = 10^{-6}.$$

by using $e^\xi < e^1 \ (\xi < 1) < 3$, we get:

$$|E_n(x)| < \frac{3}{(n+1)!} < 10^{-6}$$ (that's ok!) (we use an intermediate estimate)

$\Rightarrow (n+1)! > 3\times10^6$. it's easy to check by using calculator that $10! = 3628800 > 3\times 10^6$

so the smallest n is $n=9$.

$\Rightarrow e \approx 1 + \frac{1}{1!} + \cdots + \frac{1}{9!} = 2.718281\underset{\triangle}{5}2557\ldots$

compare with the exact value of $e = 2.718281\underset{\triangle}{8}2846\ldots$

the first 6 digits after the decimal point are the same which implies the error less than 10^{-6}.

Q2. Try to approximate $\sin x$. with $tol = 10^{-3}$.

(1) $\sin x \approx x$ just first order approximation.

so $|E(x)| = \left|\frac{\cos\xi}{3!}x^3\right| < 10^{-3}$, by using $|\cos\xi| \le 1$, we get:

$|E(x)| \le \frac{|x^3|}{6} < 10^{-3}$, $\Rightarrow |x| < 0.1817 = m$.

this m means under the error bound 10^{-3}, the first order approximation $y=x$ just allows us do the approximation when $|x|<m$. actually $|x-0|<m$ for now $x_0=0$.

(2) $\sin x \approx x - \frac{1}{3!}x^3$, second order.

now $|E(x)| = \left|\frac{\cos\xi}{5!}x^5\right| < 10^{-3}$.

$\Rightarrow |E(x)| \le \left|\frac{x^5}{5!}\right| < 10^{-3} \Rightarrow |x| < 0.6543 = M$

compare the M with m, we can see with the increasing of the order. the allowed domain for the x expanding fast. That's another point of viewing approximation.

Q3. Try to estimate $\sqrt{2}$ and deduce the error bound.

Pf: we can use $\sqrt{x}$ or $\sqrt{1+x}$. now we choose the later one.

$f(x) = \sqrt{1+x}$, we know $f(1) = \sqrt{2}$. and $f(0) = 1$, $f(3) = 2$ are easy to compute.

of course 0 is closer to 1 and more convenient to compute. So we choose $x_0 = 0$

$f(x) = \sqrt{1+x} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(\xi)}{3!}x^3$ (we choose $n=2$ for now we don't have tol, we can choose n by ourselves and get some error bound for this case)

So $f(0)=1$, $f'(x)\big|_{x=0} = \frac{1}{2}\frac{1}{(1+x)^{\frac{1}{2}}}\Big|_{x=0} = \frac{1}{2}$. , $f^{(2)}(x)\big|_{x=0} = -\frac{1}{4}\frac{1}{(1+x)^{\frac{3}{2}}}\Big|_{x=0} = -\frac{1}{4}$.

$\Rightarrow f(x) \approx 1+\frac{x}{2}-\frac{x^2}{8}$. with $E(x) = \frac{f^{(3)}(\xi)}{3!}x^3 = \frac{1}{16}\frac{1}{(1+\xi)^{\frac{5}{2}}}x^3$. $\xi \in (0,x)$

so let $x=1$. $\sqrt{2} = f(1) \approx 1+\frac{1}{2}-\frac{1}{8} = 1.375$.

And $|E(x)| = \left|\frac{1}{16}\cdot\frac{1}{(1+\xi)^{\frac{5}{2}}}\right|$, for $\xi \in (0,1)$ $\frac{1}{1+\xi} < 1$

$\Rightarrow |E(x)| < \frac{1}{16}$ $\Leftarrow$ this is an upper bound for our error.
$\approx 0.061\ldots$

compare with the exact value $\sqrt{2} = 1.414\ldots$ we know that's right.

Indefinite integral:

Like $(+,-)$, $(\times,\div)$, the $(dx, \int dx)$ are pair of inverse operation.

$dx \Rightarrow$ given a f , use the limit to find its derivative (if exists), the result is unique, comes from the uniqueness of limit;

$\int \Rightarrow$ given a g, we try to find a $F(x)$ s.t $F'(x) = g$. so we can see the integral depends on the conclusion of differential. And the result is not unique which is due to the fact that $(F(x)+C)' = g$, C is constant.

So now we would give some examples of indefinite integral which are also some basic conclusions for integral.

Q4. $\int \frac{1}{\cos^2 x \sin^2 x}dx$.

we know $1 = \cos^2 x + \sin^2 x = \sec^2 x - \tan^2 x = \csc^2 x - \cot^2 x$

now we choose first formula

$$\Rightarrow \int \frac{1}{\cos^2 x \sin^2 x}dx = \int \frac{\cos^2 x+\sin^2 x}{\cos^2 x \cdot \sin^2 x}dx = \int \frac{1}{\sin^2 x}dx + \int \frac{1}{\cos^2 x}dx$$

we have $\begin{cases} (\tan x)' = \sec^2 x = \frac{1}{\cos^2 x} \\ (-\cot x)' = \csc^2 x = \frac{1}{\sin^2 x} \end{cases} \Rightarrow \begin{array}{l} \int \frac{1}{\cos^2 x} = \tan x + C \\ \int \frac{1}{\sin^2 x} = -\cot x + C \end{array}$

$\Rightarrow$ so $\int \frac{1}{\cos^2 x \sin^2 x} dx = \tan x - \cot x + \boxed{C}$. don't forget it.

Q5. $\int \tan x \, dx$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$, we use substitution like $t = \cos x$

so $dt = -\sin x \, dx$,

$\Rightarrow \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{t} dt = -\ln|t| + C \overset{\text{back to 'x'}}{=\!=\!=} \underbrace{-\ln|\cos x| + C}$ final result.

Q6. $\int \frac{1}{a^2+x^2} dx$

$\frac{1}{a^2} \int \frac{1}{1+(\frac{x}{a})^2} dx$. (1) do the substitution again, $t = \frac{x}{a} \Rightarrow dt = \frac{1}{a} dx$.

$$(1) = \frac{1}{a^2} \int \frac{a\,dt}{1+t^2} = \frac{1}{a} \int \frac{dt}{1+t^2} = \frac{1}{a} \arctan t + C = \frac{1}{a} \arctan \frac{x}{a} + C$$

(we use $(\arctan t)' = \frac{1}{1+t^2}$ here)

Q7 $\int \frac{1}{\sqrt{a^2-x^2}} dx \quad |x| < a$

$\frac{1}{a} \cdot \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}} dx$ (2), $t = \frac{x}{a} \Rightarrow dt = \frac{1}{a} dx$

$$(2) = \frac{1}{a} \int \frac{a\,dt}{\sqrt{1-t^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin \frac{x}{a} + C \quad \left((\arcsin t)' = \frac{1}{\sqrt{1-t^2}}\right)$$

Q8. $\int \frac{x^4+1}{x^2+1} dx$.

we try to decompose it into polynomial form.

$$\frac{x^4+1}{x^2+1} = \frac{x^2(x^2+1) - x^2 - 1 + 1}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$$

integration both sides :

$$\int \frac{x^4+1}{x^2+1} dx = \int x^2 dx - \int 1 \, dx + \int \frac{1}{x^2+1} dx = \frac{1}{3}x^3 - x + \arctan x + C$$

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* Last time: (1) write down degree k Taylor polynomial of $f(x)$ centered at c

  (2) Approximate $f(a)$ up to k-th decimal places.

* One more example on approximating function value using Taylor polynomial.

Let $f(x)=\sqrt{4+x}$, $a=0.1$, using Taylor polynomial of degree 3 to approximate $f(a)$ and state the maximal possible error.

① Examine the derivatives of $f(x)$

$$f(x)=\sqrt{4+x}=(4+x)^{\frac{1}{2}}$$

$$f'(x)=\frac{1}{2}(4+x)^{-\frac{1}{2}} \qquad f''(x)=-\frac{1}{4}(4+x)^{-\frac{3}{2}}$$

$$f'''(x)=\frac{3}{8}(4+x)^{-\frac{5}{2}} \qquad f^{(4)}(x)=-\frac{15}{16}(4+x)^{-\frac{7}{2}}$$

② Choose a center c $\begin{cases} |c-a|<1 \\ f^{(n)}(c) \text{ is easy to compute} \end{cases}$

choose $c=0$ s.t. $(4+c)^{\frac{k}{2}}=2^k$

③ Write down the Taylor polynomial $P_3(x)$

$$P_3(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)(x-c)^3}{6}$$

$$= 2 + \frac{1}{2}\cdot 2^{-1}\cdot x - \frac{1}{2}\cdot\frac{1}{4}2^{-3}\cdot x^2 + \frac{1}{6}\cdot\frac{3}{8}2^{-5}x^3$$

$$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$$

So approximate value $P_3(a) = 2 + \frac{1}{4}\cdot 0.1 - \frac{1}{64}(0.1)^2 + \frac{1}{512}(0.1)^3$

$$= 2.0248$$

④ Estimate maximal possible error

$$E_3(x) = \frac{f^{(4)}(\xi)(x-c)^4}{4!} = -\frac{1}{4!}\frac{15}{16}(4+\xi)^{-\frac{7}{2}}x^4$$

Maximal possible error is an upper bound for $|E_4(a)|$

$$|E_3(a)| = \frac{1}{4!}\frac{15}{16}(0.1)^4\,|4+\xi|^{-\frac{7}{2}}, \quad \xi \text{ between } \underset{0.1}{\overset{\|}{a}} \text{ and } \underset{0}{\overset{\|}{c}}$$

Note $x^{-\frac{7}{2}} \searrow$ as $x \nearrow$

So $|4+\xi|^{-\frac{7}{2}} \le |4+0|^{-\frac{7}{2}} = 2^{-7}$

So maximal possible error

$$= \frac{1}{4!}\frac{15}{16}(0.1)^4\cdot 2^{-7} = 3.0517578 \times 10^{-8}$$

Check error $= \sqrt{4+0.1} - P_3(0.1) = 2.99933416 \times 10^{-8}$

real error $\le$ maximal possible error.

<u>Remark</u>: When you are asked to approximate $f(a)$ up to 3-decimal places, you are actually requiring that

$$|E_n(a)| \le \underset{L(n)}{\underbrace{\text{maximal possible error}}} < 10^{-3}$$

$L(n)$ we talked about last time

* Use Taylor thm to do proofs:

Eg. Show that $x + \frac{x^3}{3} \le \tan x$, $0 < x < \frac{\pi}{2}$

Consider degree 3 taylor polynomial of $\tan x$ at center 0

$\tan 0 = 0$,

$\tan' x = \sec^2 x$, $\tan'(0) = 1$

$\tan'' x = 2\sec^2 x \tan x$, $\tan''(0) = 0$

$\tan'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$, $\tan'''(0) = 2$

$\tan^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

So $P_3(x) = x + \frac{x^3}{3}$,

$\tan x - (x + \frac{x^3}{3}) = E_3(x) = \frac{\tan^{(4)}(\xi)\, x^4}{4!}$, for some $\xi \in (0, x)$

Since $\xi \in (0, x)$, i.e. $\xi \in (0, \frac{\pi}{2})$.

$\sec \xi > 0$, $\tan \xi > 0$

$\Rightarrow$ $\tan^{(4)}(\xi) > 0$. $\Rightarrow$ $E_3(x) \geq 0$.

$\Rightarrow$ $\tan x \geq x + \frac{x^3}{3}$

* Indefinite integral

Recall inverse function f^{-1} of f is defined by

$$(f^{-1} \circ f)(x) = x$$

If one think of $'$ and $\int$ are actions on functions which are inverse to each other :

$$\int f'(x)\,dx = f(x) + C$$

any constant

Eg. $(\sin x)' = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

Remark : NOT all functions are integrable , i.e. not all functions are derivatives of other functions.

Eg. Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

Ex : Approximate $f(x) = \tan^{-1} x$, $a = \frac{1}{2}$.

w/ Taylor polynomial of order 3.

Estimate the maximal possible error

6/11/2014 Math 1010C Tutorial - 9

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PLAN: $\begin{cases} \cdot \text{ Taylor's theorem;} \\ \cdot \text{ trig. } f'^n \text{ (math 1010C webpage);} \end{cases}$

Review: Taylor's theorem

Taylor's theorem (Peano remainder)

$f: \mathbb{R} \to \mathbb{R}$, n-times differentiable at $c \in \mathbb{R}$, then there exists a function $h_n(x): \mathbb{R} \to \mathbb{R}$, s.t.

h(x) (x-c)^n

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{\overset{!!}{T_n(x)} \text{ Taylor polynomial of order } n} + \underbrace{h_n(x)(x-c)^n}_{R_n(x) \text{ Remainder term "approximation error"}}$$

& $\underbrace{\lim_{x \to c} h_n(x) = 0}$;

$\Leftrightarrow \lim_{x \to c} \frac{R_n(x)}{(x-c)^n} = 0$; Remainder of Peano form.

($\Leftrightarrow R_n(x) = o(|x-c|^n)$, small o notation)

Taylor's theorem (Mean-value form, Lagrangian form of remainder)

$f: (a,b) \to \mathbb{R}$, $c \in (a,b)$ fixed. $\forall x \overset{<\text{or}>}{\neq} c \in (a,b)$, if f is $(n+1)$-times differentiable on $\overset{x<c}{(x,c)}$ or $\overset{x>c}{(c,x)}$, & $f^{(n)}$ is continuous on $\begin{cases} [x,c], & x<c; \\ [c,x], & x>c; \end{cases}$

Then $f(x) = \underbrace{T_n(x)}_{\text{Taylor polynomial as above}} + \underbrace{R_n(x)}_{\text{Remainder}}$;

$\exists \xi \in \begin{cases} (x,c), & x<c \\ (c,x), & x>c \end{cases}$, s.t. $R_n(x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}}_{\text{Remainder of Lagrangian form.}}$;

[Rmk: Cauchy form $R_n(x) = \frac{f^{(n+1)}(\xi_c)}{n!}(x-\xi_c)^n(x-c)$;]

Focus On: (a) Computation of Taylor polynomial;

(b) Evaluating limits (Peano form);

(c) Approximating f'ns (Mean-Value form);

a). ExD. (Famous ones)

$R_n(x)$ are some f'ns s.t $\lim_{x\to 0} \frac{R_n(x)}{x^n} = 0$

1) $f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n^f(x)$,

2) $g(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + R_{2m}^g(x)$;

3) $h(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m+1}^h(x)$,

4) $k(x) = (1+x)^m$, $m \notin \mathbb{Z}_{\geq 0}$ (eg: $m = -1, \frac{1}{2}, -\frac{1}{2}$)

5) $l(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + R_n^l(x)$;

6) $p(x) = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{m-1}\frac{x^{2m-1}}{2m-1} + R_{2m}^p(x)$;

7) $q(x) = \tan x$ (HARD!)

$= x + \frac{x^3}{3} + 0 + R_4^q(x)$;

(above $R_n(x)$'s are all different!)

Sol'n: 1) $f^{(k)}(x) = e^x$, $k = 1, 2, \cdots$; $f(0) = 1$, $f^{(k)}(0) = 1$; Hence done;

2) $g(x) = \sin x$, $g^{(k)}(x) = \sin(x + k\cdot\frac{\pi}{2})$, at $x = 0$

$g(0) = 0$, $g^{(2m)}(0) = \sin(m\pi) = 0$;

$g^{(2m-1)}(0) = \sin(m\pi - \frac{\pi}{2}) = (-1)^{m-1}$ $(m = 1, 2, \cdots)$ done

3) $h(x) = \cos x$. $h^{(k)}(x) = \cos(x + k\cdot\frac{\pi}{2})$;

$h(0) = 1$. $h^{(2m)}(0) = (-1)^m$, $h^{(2m-1)}(0) = 0$, $(m = 1, 2, \cdots$

4) $k(x) = (1+x)^m$, $(k(x))^{(n)} = m\cdot(m-1)\cdots(m-n+1)(1+x)^{m-n}$

m-n

$k(0) = 1$, $k^{(n)}(0) = m(m-1)\cdots(m-n+1)$;

Hence

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + R_n(x);$$

Let $n=2$, $m=-1, \frac{1}{2}, -\frac{1}{2}$, then

$$\frac{1}{1+x} = 1 - x + x^2 + R_2(x);$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + R_2(x);$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + R_2(x); \quad \text{etc};$$

5) $l(x) = \ln(1+x)$; $\quad l'(x) = \frac{1}{1+x}$; $\quad l'' = -\frac{1}{(1+x)^2}$;

$$l^{(k)}(x) = (-1)^{k-1}\cdot\frac{(k-1)!}{(1+x)^k};$$

$l(0) = 0$, $\quad l^{(k)}(0) = (-1)^{k-1}\cdot(k-1)!$; $\quad$ done;

6) $p(x) = \arctan x$; $\quad p'(x) = \frac{1}{1+x^2}$; $\quad$ Ex: $\begin{cases} p^{(2m)}(0) = 0; \\ p^{(2m-1)}(0) = (-1)^{m-1}\cdot(2m-2)! \end{cases}$

7) $q(x) = \tan x$; $\quad q'(x) = \frac{1}{\cos^2 x}$, $\quad q''(x) = \frac{2\sin x}{\cos^3 x}$;

$q'''(x) = 2\cdot\frac{1+2\sin^2 x}{\cos^4 x}$; $\quad q^{(4)}(x) = 8\sin x\,\frac{2+\sin^2 x}{\cos^5 x}$;

$q(0) = 0$, $\quad q'(0) = 1$, $\quad q''(0) = 0$, $\quad q'''(0) = 2$, $\quad q^{(4)} = 0$;

Ex 1. $f(x) = x^3 + 3x + 1$ up to order 4 at $x=0$ & $x=1$.

Sol'n: $c=0$. $\quad f'(x) = 3x^2+3$, $\quad f''(x) = 6x$, $\quad f'''(x) = 6$, $\quad f^{(4)}(x) = 0$;

$\quad x=0$: 3 $\quad x=0$: 0 $\quad x=0$: 6 $\quad x=0$: 0

$f(0)=1 \Rightarrow$

$$f(x) = 1 + 3\cdot(x-0) + \frac{0}{2}\cdot(x-0)^2 + \frac{6}{3!}\cdot(x-0)^3 + 0\cdot(x-0)^4 + R_4$$

$$= 1 + 3x + x^3 + R_4(x).$$

(Hence actually $R_4(x) \equiv 0$).

$c=1$ $\quad f(1) = 5$, $f'(1) = 6$, $f''(1) = 6$, $f'''(1) = 6$, $f^{(4)}(1) = 0$

$$f(x) = 5 + 6(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 + 0 + \widetilde{R}_4(x)$$

$$= 5 + 6(x-1) + 3(x-1)^2 + (x-1)^3 + \underbrace{\widetilde{R}_4(x)}_{\parallel} ; \qquad \square$$

Rmk: • $R_4(x) \equiv 0$, $\widetilde{R}_4(x) \equiv 0$; $\qquad \frac{\overset{0}{f^{(4)}}(\xi)}{4!}(x-1)^4 \equiv 0$;

• In general, let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$;

$$\begin{cases} p'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}; \\ p''(x) = 2a_2 + 6a_3 x + \cdots + (n-1)n\, a_n x^{n-2}; \\ \vdots \\ p^{(n)}(x) = 1 \cdot 2 \cdot \cdots \cdot n \cdot a_n = n!\, a_n; \\ p^{(n+1)}(x) = 0 = p^{(n+i)}(x), \quad i = 1, 2, \ldots \end{cases}$$

at $c=0$ $\qquad p^{(k)}(0) = k!\, a_k$, i.e. $a_k = \frac{p^{(k)}(0)}{k!}$;

Hence $p(x) = p(0) + \frac{p'(0)}{1!}x + \frac{p''(0)}{2!}x^2 + \frac{p'''(0)}{3!}x^3 + \cdots + \frac{p^{(n)}(0)}{n!}x^n$;

at $c=x_0$ $\quad$ If write $p(x)$ as (substitue $y = x - x_0$, $x = x_0 + y$ in $p(x)$),

$$p(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + A_n(x-x_0)^n;$$

Then $A_0 = p(x_0)$, $A_1 = \frac{p'(x_0)}{1!}$, $A_2 = \frac{p''(x_0)}{2!}$, $\ldots$, $A_n = \frac{p^{(n)}(x_0)}{n!}$;

[**Reason:** write $p(x) = p(x_0) + p'(x_0)(x-x_0) + \cdots + \frac{p^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$;
use Lag. form remainder $R_n(x) = \frac{p^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$; but $p^{(n+1)}(x) \equiv 0$;
$\Rightarrow R_n(x) \equiv 0$. done.] $\square$

Ex 2: $f(x) = e^{\sin x}$ up to order 3;

Sol'n: $f(0) = 1$; $f'(x) = (\cos x) \cdot e^{\sin x}$; $f'(0) = 1$,

$f''(x) = (-\sin x) \cdot e^{\sin x} + \cos^2 x \cdot e^{\sin x}$; $f''(0) = 1$,

$f'''(x) = -\cos x \cdot e^{\sin x} - \sin x \cos x \, e^{\sin x} + 2\cos x \cdot \sin x \, e^{\sin x} + \cos^3 x \cdot e^{\sin x}$;

$f'''(0) = 0$;

Hence $f(x) = 1 + x + \frac{1}{2}x^2 + 0 + R_3(x^3)$;

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b). Ex 3. (Revision Exercise 2. 11(i), revised)

$$\lim_{x \to 0} \frac{24 - 24\cos(x) - 12x^2 + x^4}{\sin^6(x)};$$

Sol'n: since $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + R_6(x)$;

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ $\qquad \lim_{x \to 0} \frac{R_6(x)}{x^6} = 0$;

Hence $24\cos(x) = 24 - 12x^2 + x^4 - \frac{x^6}{30} + 24 R_6(x)$;

Now $\underbrace{\frac{24 - 24\cos x - 12x^2 + x^4}{\sin^6 x}} = \frac{\frac{x^6}{30} - 24 R_6(x)}{\sin^6 x} = \frac{1}{30} \cdot \frac{x^6}{\sin^6 x} - \frac{24 R_6}{\sin^6 x}$

But $\frac{R_6(x)}{x^6} \to 0$; Hence $\frac{R_6(x)}{\sin^6 x} = \frac{R_6(x)}{x^6} \cdot \frac{x^6}{\sin^6 x} \to 0 \cdot 1 = 0$;

Hence $\lim_{x \to 0} (\;\text{''}\;) = \frac{1}{30}$; $\square$

Ex 4. (Revision Exercise 2: 13(f))

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x};$$

Sol'n: use $\sin x = x-\frac{x^3}{6}+\underbrace{R_3(x)}\ (\Leftrightarrow \overbrace{h_3(x)}\cdot x^3)$;

$\lim_{x\to 0} h_3(x)=0.$

$\lim_{x\to 0}\frac{R_3(x)}{x^3}=0;$

$$\sin^2 x = \left(x-\frac{x^3}{6}+h(x)\cdot x^3\right)^2 = x^2-\frac{x^4}{3}+\underbrace{\frac{x^6}{36}+2\left(1-\frac{x^2}{6}\right)h(x)\cdot x^4}_{g(x)\cdot x^4}$$

where $\lim_{x\to 0} g(x)=0$

Now $x^2-\sin^2 x = \frac{x^4}{3}-g(x)\cdot x^4$

$$\lim_{x\to 0}\frac{x^2-\sin^2 x}{\sin^2 x\cdot x^2}=\lim_{x\to 0}\frac{\frac{x^4}{3}-g(x)\cdot x^4}{x^4}\cdot\frac{x^2}{\sin^2 x}=\lim_{x\to 0}\left(\frac{1}{3}-g(x)\right)\cdot\frac{x^2}{\sin^2 x}$$

$$=\frac{1}{3};$$

(c). Lagrangian form of remainder can be used to evalute the approximation error using $f^{(n+1)}(x)$ information, hence $R_n(x)$ can be controlled by $f^{(n+1)}(x)$;

e.g: $f(x)=\sin(x)$, then $f^{(n)}(x)=\sin\left(x+\frac{n}{2}\pi\right)$, $|f^{(n)}(x)|\le 1$;

then $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+(-1)^m\frac{x^{2m+1}}{(2m+1)!}+\underbrace{R_{2m+2}(x)}_{\text{remainder}}$;

$$R_{2m+2}(x)=\frac{f^{(2m+3)}(\xi)}{(2m+3)!}(x-0)^{2m+3};\quad \xi\in(0,x)\text{ or }(x,0)$$

$$\underbrace{|R_{2m+2}(x)|\le\frac{|x|^{2m+3}}{(2m+3)!}}_{\text{remainder controlled by this}};$$

e.g using $\sin x \approx x-\frac{x^6}{6}$, then error $|R_4(x)|\le\frac{x^5}{120}$

x<0.4129.... $\approx 23.5^\circ$, error < 0.0001, etc.

Now if $\frac{x^5}{120} < 0.001 \iff x < 0.6544\ldots\ (\approx 37.5^\circ)$

the error of using $x - \frac{x^6}{6}$ approximate $\sin x$ is < 0.001;

e.g using $\sin x \approx x$, then error $|R_2(x)| \leq \frac{x^3}{6}$;

if $\frac{x^3}{6} < 0.001 \iff x < 0.1817\ldots\ (\approx 10^\circ)$.

the error of using x approximating $\sin x$ is < 0.001;

$\square$

EX5. If $f(x)$ is twice-differentiable on $[0, 1]$, & for $0 \leq x \leq 1$, $|f(x)| \leq 1$, $|f''(x)| < 2$; Prove when $0 \leq x \leq 1$, $|f'(x)| \leq 3$;

Pf: Using Taylor thm w/ Lagrangian form of Remainder.

$$\begin{cases} f(1) = f(x) + f'(x)(1-x) + \frac{f''(\xi)}{2}(1-x)^2; & \exists \xi \text{ between } 1 \,\&\, x, \\ f(0) = f(x) + f'(x)(0-x) + \frac{f''(\eta)}{2}(-x)^2; & \exists \eta \text{ between } 0 \,\&\, x. \end{cases}$$

$$f(1) - f(0) = f'(x) + \frac{1}{2} f''(\xi)(1-x)^2 - \frac{1}{2} f''(\eta)\, x^2;$$

$$\Rightarrow |f'(x)| \leq |f(1)| + |f(0)| + \frac{1}{2}|f''(\xi)|(1-x)^2 + \frac{1}{2}|f''(\eta)| \cdot x^2$$

$$\leq 2 + (1-x)^2 + x^2 \leq 2 + 1 = 3.$$

$\square$

TA9 (c

Tutorial 9

Topics : Taylor's Theorem & Introduction to Indefinite Integral.

Q1) Suppose the taylor's expansion of f, g are

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \quad ; \quad g(x) = b_1x + b_2x^2 + b_3x^3 + \cdots$$

a) find taylor expansion of $f \circ g$ up to degree 3

b) find first 3 derivatives of $f \circ g$ at $x = 0$.

Q2)

Given $f : \mathbb{R} \to \mathbb{R} \; ; \; x \longmapsto \begin{cases} e^{-x^{-2}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Show that taylor series of f is identically 0

[In order words show that $f^{(k)}(0) = 0 \quad \forall k = 0, 1, 2, \cdots$]

Q3) Given $f(x) = \ln(x+2)$, Let $k \in \mathbb{N}$, $x > -2$

a) find taylor polynomial of f up to degree k at center $x_0 = 0$.

b) Suggest an upper bound for the error term at $x = 1$.

c) Suggest number of terms we need to expand s.t. the error of $f(1)$ is less than 10^{-3}.

Q4) Compute the following indefinite integral

a) $\int x^k \, dx$, $k \in \mathbb{N}$ b) $\int (x^3+1)^2 \, dx$

c) $\int \frac{x+1}{\sqrt{x}} \, dx$

Recall:

Taylor's Thm :

Let $k \in \mathbb{N}$, If $f: \mathbb{R} \to \mathbb{R}$ smooth then $f(x) = P_k(x) + R_k(x)$ s.t.

$$\begin{cases} P_k(x) = f(x_0) + \dots + \dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k \\ R_k(x) = \dfrac{f^{(k+1)}(\xi_x)}{(k+1)!}(x-x_0)^{k+1} \text{ for some } \xi_x \text{ between } x, x_0. \end{cases}$$

[Lagrange form of remainder]

Indefinite Integral

- Given a function $f(x)$, an anti-derivative of $f(x)$ is any function $F(x)$ s.t. $F'(x) = f(x)$ [on suitable domains]

- If F is anti-derivative of $f(x)$ then $\int f(x)\,dx = F(x) + c$ for some $c \in \mathbb{R}$ is called the indefinite integral of $f(x)$.

Sol^n

Q1) a) $f \circ g(x) = f(b_1x + b_2x^2 + b_3x^3 + \dots)$

$= a_0 + a_1(b_1x + b_2x^2 + b_3x^3 + \dots) + a_2(b_1x + b_2x^2 + b_3x^3 + \dots)^2$

$+ a_3(b_1x + b_2x^2 + b_3x^3 + \dots)^3 + \dots$

$= (a_0) + (a_1b_1)x + (a_1b_2 + a_2b_1)x^2$

$+ (a_1b_3 + 2a_2b_1b_2 + a_3b_1^3)x^3$ + higher order terms

b)
$$\begin{cases} f \circ g(0) = (0!)(a_0) = a_0 \\ (f \circ g)'(0) = (1!)(a_1b_1) = a_1b_1 \\ (f \circ g)''(0) = (2!)(a_1b_2 + a_2b_1) = 2a_1b_2 + 2a_2b_1 \\ (f \circ g)'''(0) = (3!)(a_1b_3 + 2a_2b_1b_2 + a_3b_1^3) = 6a_1b_3 + 12a_2b_1b_2 + 6a_3b_1^3 \end{cases}$$

Q2) Given $f : \mathbb{R} \to \mathbb{R}$; $x \longmapsto \begin{cases} e^{-\frac{1}{x^2}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ (Prove by M.I.)

Claim : $\forall n \in \mathbb{N}$, $f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ for some polynomial P_n .

- for $n = 0$, the claim is true .
- Assume that the claim is true for $n = k$.

i.e. $f^{(k)}(x) = \begin{cases} P_k\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Notice that $f^{(k+1)}(x) = f^{(k)\prime}(x)$

for $x \neq 0$, $\quad f^{(k+1)}(x) = \left(P_k(\frac{1}{x}) e^{-\frac{1}{x^2}} \right)'$

$$= P_k'(\tfrac{1}{x})\left(\frac{-1}{x^2}\right)e^{-\frac{1}{x^2}} + P_k(\tfrac{1}{x})\left(\frac{2}{x^3}\right)e^{-\frac{1}{x^2}}$$

$$= \left[P_k'(\tfrac{1}{x})\left(\frac{-1}{x^2}\right) + P_k(\tfrac{1}{x})\left(\frac{2}{x^3}\right) \right] e^{-\frac{1}{x^2}} =: P_{k+1}(\tfrac{1}{x})\, e^{-\frac{1}{x^2}}$$

for $x = 0$, $\quad f^{(k+1)}(0) = \lim_{x\to 0} \dfrac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x \to 0} (\tfrac{1}{x}) P_k(\tfrac{1}{x})\, e^{-\frac{1}{x^2}}$

$$= \lim_{y\to\pm\infty} \frac{y P_k(y)}{e^{y^2}} \overset{\left(\frac{\pm\infty}{\pm\infty}\right)}{=} \lim_{y\to\pm\infty} \frac{(yP_k(y))'}{-2y\, e^{-y^2}}$$

$=$ ... perform finitely number of times of L'Hôpital Rule ... $= 0$

Hence

$$f^{(k+1)}(x) = \begin{cases} P_{k+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

By MI the claim is true

By Taylor's Thm.

$$f(x) = f(0) + \cdots + \frac{f^{(k)}(0)}{k!} x^k + \cdots$$

$$= 0 + 0x + 0x^2 + \cdots + 0x^k + \cdots$$

//

Q3) Given $f(x) = \ln(x+2)$, $x > -2$; For any $k \in \mathbb{N}$.

a) By taylor's thm, $P_k(x) = f(0) + f'(0)x + \cdots + \frac{f^{(k)}(0)}{k!} x^k$

where
$$f(0) = \ln(x+2)\Big|_{x=0} = \ln 2$$
$$f'(0) = \frac{1}{(x+2)}\Big|_{x=0} = \frac{1}{2}$$
$$f''(0) = \frac{(-1)}{(x+2)^2}\Big|_{x=0} = \frac{-1}{4}$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(k)}(0) = \frac{(-1)^{k-1}(k-1)!}{(x+2)^k}\Big|_{x=0} = \frac{(-1)^{k-1}(k-1)!}{2^k}$$

hence
$$P_k(x) = \ln 2 + \sum_{i=1}^{k} \left(\frac{(-1)^{i-1}(i-1)!}{2^i}\right)\left(\frac{x^i}{i!}\right) = \ln 2 + \sum_{i=1}^{k}\left(\frac{(-1)^{i-1}}{i 2^i} x^i\right)$$

b) By taylor's thm, $R_k(x) = \dfrac{f^{(k+1)}(\xi_x)}{(k+1)!}(x)^k$ for some ξ_x between 0 and x.

$$\Rightarrow \left| f(1) - P_k(1) \right| = \left| R_k(1) \right| = \left| \frac{f^{(k+1)}(\xi_1)}{(k+1)!}(1)^k \right| \quad [\xi_x \text{ depends on } x]$$

$$= \left| \frac{(-1)^k (k)!}{(\xi_1 + 2)^{k+1}} \left(\frac{1}{(k+1)!}\right) \right| = \frac{1}{k+1} \left| \frac{1}{\xi_1 + 2} \right|^{k+1} \quad \text{where } 0 < \xi_1 < 1$$

$$\le \frac{1}{k+1}\left(\frac{1}{2}\right)^{k+1}$$

c) If $\frac{1}{k+1}\left(\frac{1}{2}\right)^{k+1} < 10^{-3}$ then $|f(1) - P_k(1)| < 10^{-3}$

try $k = 7 \Rightarrow \frac{1}{k+1}\left(\frac{1}{2}\right)^{k+1} = \frac{1}{2048} < \frac{1}{1000} = 10^{-3}$

Hence after expanding f up to 7^{th} order then the taylor polynomial approximate at $x=1$ will has error less than 10^{-3}.

//

Q4) a) $\int x^k \, dx = \frac{x^{k+1}}{k+1} + c \quad \exists c \in \mathbb{R}.$

b) $\int (x^3+1)^2 \, dx = \int x^6 + 2x^3 + 1 \, dx$

$= \frac{x^7}{7} + \frac{x^4}{2} + x + c \quad \exists c \in \mathbb{R}$

c) $\int \frac{x+1}{\sqrt{x}} \, dx = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}} \, dx$

$= \frac{2x^{\frac{3}{2}}}{3} + 2x^{\frac{1}{2}} + C \quad \exists c \in \mathbb{R}.$